# Mathematics: analysis and approaches
## Higher level
## Paper 1

Name

worked solutions

Date: ____________________

2 hours

SOLUTION KEY

**Instructions to candidates**

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written in the answer boxes provided.
- Section B: answer all questions on the answer sheets provided. Write your name on each answer sheet and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

***exam: 11 pages***

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A (56 marks)

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 6]

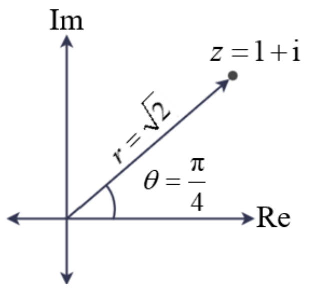
Consider the complex number $z=1+i$.

(a) Express $z$ in modulus-argument form $r\,\text{cis}\,\theta$. [3]

(b) Hence, find $z^9$ and express it in Cartesian form $a+bi$. [3]

**Solution**:

(a) $z=\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$




(b) apply De Moivre's theorem

$$z^9=\left[\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)\right]^9=\left(\sqrt{2}\right)^9\text{cis}\left(\frac{9\pi}{4}\right)$$

$$=\left(2^{\frac{1}{2}}\right)^8\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$$

$$=16\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$$

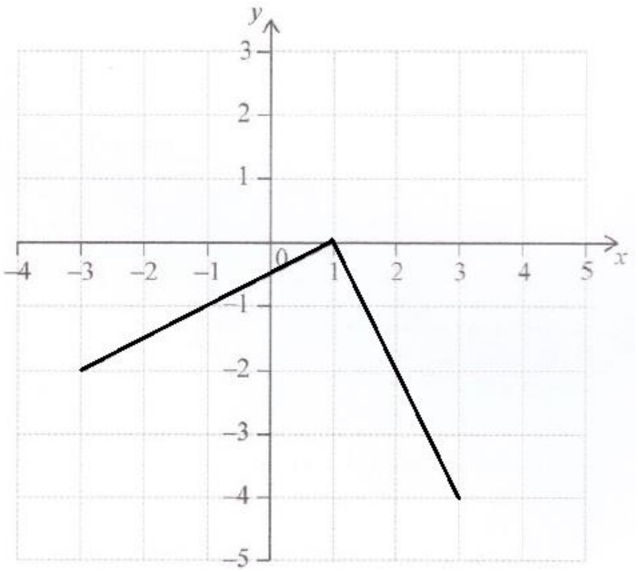
$$=16\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+\text{i}\sin\left(\frac{\pi}{4}\right)\right]$$

$$=16\sqrt{2}\left(\frac{\sqrt{2}}{2}\right)+16\sqrt{2}\left(\frac{\sqrt{2}}{2}\right)\text{i}$$

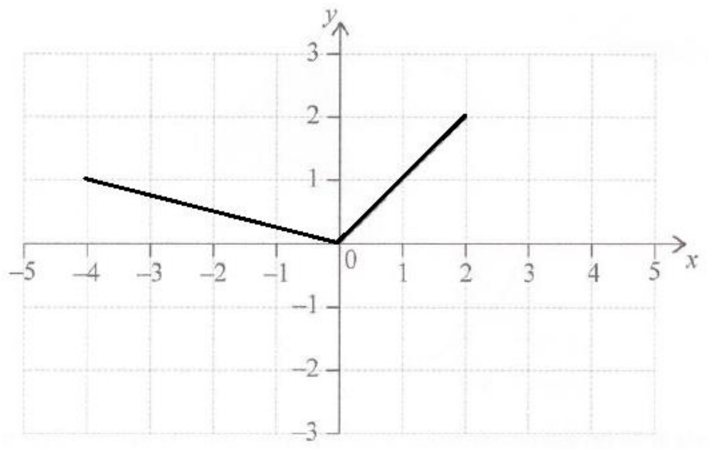
$$z^9=16+16\text{i}$$

**2.** [Maximum mark: 4]

Below is the graph of the function $g$ for $-3 \le x \le 3$.



Another function, $h$, can be written in the form $h(x) = a\left[g(x+b)\right]$. The graph of $h$ is shown below. Write down the value of $a$ and the value of $b$.



**Solution**:

The graph of $h$ is produced by performing the following three transformations on the graph of $g$:

(i) reflection about the $x$-axis ($a$ is negative); (ii) vertical shrink by a factor of $\frac{1}{2}$ $\left(|a| = \frac{1}{2}\right)$; and

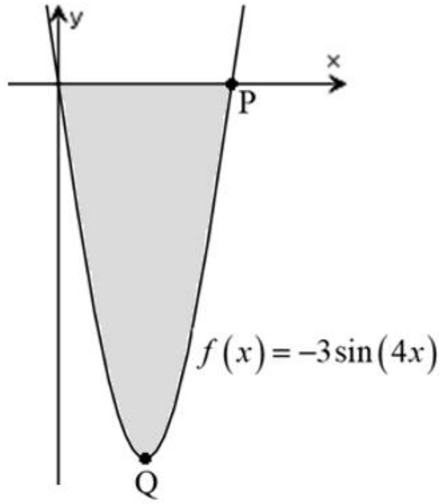
(iii) translation one unit to the left $(b = 1)$.

Therefore, $a = -\frac{1}{2}$ and $b = 1$

**3.** [Maximum mark: 7]

A portion of the graph of $f(x) = -3\sin(4x)$ is shown.

The point P is an $x$-intercept with coordinates $(p, 0)$.




(a) Find the value of $p$. [2]

(b) The point Q is a minimum. Write down the coordinates of Q. [2]

(c) Find the area of the shaded region that is bounded by $f$ and the $x$-axis. [3]

**<u>Solution</u>:**

(a) $-3\sin(4p) = 0 \quad \Rightarrow \quad 4p = k\pi, k \in \mathbb{Z} \quad \Rightarrow \quad p = \dfrac{\pi}{4}$

(b) due to symmetry of the graph of $f$, the $x$-coordinate of Q must be one-half the $x$-coordinate of P;

and since the maximum value of $\sin(4x)$ is 1, then the minimum value of $-3\sin(4x)$ is $-3$

therefore, the coordinates of Q are $\left(\dfrac{\pi}{8}, -3\right)$

(c) area of shaded region $= \left| \int_0^{\frac{\pi}{4}} (-3\sin(4x))\,dx \right|$

$$= \left| \left[ \frac{3}{4}\cos(4x) \right]_0^{\frac{\pi}{4}} \right| = \left| \frac{3}{4}\left[\cos(\pi) - \cos(0)\right] \right| = \left| \frac{3}{4}[-1-1] \right|$$

thus, area of shaded region $= \dfrac{3}{2}$ units$^2$

**4.** [Maximum mark: 6]

The sum of the first three terms of an arithmetic sequence is 6 and the fourth term is 16.

Find the first term, $u_1$, and the common difference, $d$, of the sequence.

**<u>Solution</u>:**

$u_1 + (u_1 + d) + (u_1 + 2d) = 6 \quad \Rightarrow \quad 3u_1 + 3d = 6$ and $u_1 + 3d = 16 \quad \Rightarrow \quad u_1 = 16 - 3d$

Substituting gives $3(16 - 3d) + 3d = 6 \quad \Rightarrow \quad 48 - 6d = 6 \quad \Rightarrow \quad 6d = 42 \quad \Rightarrow \quad d = 7$

$u_1 = 16 - 3(7) = -5$ <u>answers</u>: $u_1 = -5$ and $d = 7$

**5.** [Maximum mark: 7]

(a) Given $f(x)=x^2+4x-10,\ x\le -2$ show that $f^{-1}(x)=-2-\sqrt{x+14},\ x\ge -14$. [4]

(b) The graphs of $f$ and $f^{-1}$ intersect at point C. Find the coordinates of C. [3]

**Solution**:

(a) $f(x)=x^2+4x-10,\ x\le -2 \quad\Rightarrow\quad y=x^2+4x-10$ switch domain & range: $x=y^2+4y-10$

solve for $y$ using 'completing the square': $x=y^2+4y-10 \Rightarrow x=y^2+4y+4-14 \Rightarrow x+14=(y+2)^2$

$y+2=\pm\sqrt{x+14} \quad\Rightarrow\quad y=-2-\sqrt{x+14}$ (negative square root because range of $f^{-1}$ is $y\le -2$)

thus, $f^{-1}(x)=-2-\sqrt{x+14},\ x\ge -14$ ***Q.E.D.***

(b) point C must be on the line $y=x$; hence, solve $x^2+4x-10=x$

$$x^2+3x-10=0 \quad\Rightarrow\quad x=\frac{-3\pm\sqrt{3^2-4(1)(-10)}}{2}=\frac{-3\pm\sqrt{49}}{2}=\frac{-3\pm 7}{2} \quad\Rightarrow\quad x=2 \text{ or } x=-5$$

$x\ne 2$ because the domain of $f$ is $x\le -2$; hence, $x=-5$ and since C is on $y=x$ then $y=-5$

thus, the coordinates of C are $(-5,-5)$

**6.** [Maximum mark: 6]

Show that $\log_2\sqrt{8}+\log_b\sqrt{ab}=\dfrac{\ln(ab^4)}{\ln(b^2)}$

**Solution**: $\log_2\sqrt{8}+\log_b\sqrt{ab}=\log_2\left[\left(2^3\right)^{\frac{1}{2}}\right]+\log_b\left(\sqrt{a}\sqrt{b}\right)$

$$=\log_2\left(2^{\frac{3}{2}}\right)+\log_b\left(a^{\frac{1}{2}}\right)+\log_b\left(b^{\frac{1}{2}}\right)$$

$$=\frac{3}{2}+\frac{1}{2}\log_b a+\frac{1}{2}$$

$$=2+\frac{1}{2}\log_b a$$

$$=\frac{4\ln b}{2\ln b}+\frac{1}{2}\cdot\frac{\ln a}{\ln b}$$

$$=\frac{4\ln b+\ln a}{2\ln b}$$

$$=\frac{\ln(b^4)+\ln a}{\ln(b^2)}$$

$$=\frac{\ln(ab^4)}{\ln(b^2)}$$ ***Q.E.D.***

**7.** [Maximum mark: 6]

When the expression $3x^3-6x^2+ax-1$ is divided by $(x+1)$ it produces the same remainder as when the expression is divided by $(x-3)$. Find the value of $a$.

**Solution**:

Let $f(x)=3x^3-6x^2+ax-1$. The remainder when $f$ is divided by $(x-c)$ is equal to $f(c)$.

$f(-1)=3(-1)^3-6(-1)^2+a(-1)-1=-3-6-a-1=-a-10$

$f(3)=3(3)^3-6(3)^2+a(3)-1=81-54+3a-1=3a+26 \quad \Rightarrow \quad 3a+26=-a-10 \quad \Rightarrow \quad 4a=-36$

thus, $a=-9$

**8.** [Maximum mark: 7]

Consider the following system of linear equations representing three planes.

A solution to the system is an ordered triple $(x,y,z)$ – a point in space.

$$\begin{aligned} 2x-y+2z&=1 \\ x+y-2z&=2 \\ x-2y+4z&=-1 \end{aligned}$$

(a) Show, with justification, that the system has an infinite number of solutions. [4]

(b) Find the equation of the line, in parametric form, on which the solutions lie. [3]

**Solution**:

(a) (R3 – R2)/3→R3 (2R2 – R1)/3→R2 R3 – R2→R3

$$\left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 1 & 1 & -2 & 2 \\ 0 & -1 & 2 & -1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

bottom row of zeros shows that an infinite number of ordered triples $(x,y,z)$ solve the system

(b) from row 2 in the last matrix above: $0x+y-2z=1$

If $z=\lambda$, then $y=2\lambda+1$

and from row 1: $2x-y+2z=1 \Rightarrow 2x-(2\lambda+1)+2\lambda=1 \Rightarrow 2x=2 \Rightarrow x=1$

thus, the parametric equation of the line on which the infinite solutions to the system lie is

$x=1, \; y=2\lambda+1, \; z=\lambda$

**9.** [Maximum mark: 7]

Find the value of $\int_0^{\frac{1}{2}} \frac{14x+1}{2x^2-x-1}\,\mathrm{d}x$ giving your answer in the form $a\ln b$ where $a,b\in\mathbb{Z}$.

**<u>Solution</u>**:

$$\frac{14x+1}{2x^2-x-1}=\frac{14x+1}{(2x+1)(x-1)}=\frac{\mathrm{A}}{2x+1}+\frac{\mathrm{B}}{x-1} \qquad \text{partial fractions}$$

$$14x+1=\mathrm{A}(x-1)+\mathrm{B}(2x+1)$$

Let $x=-\frac{1}{2}$: $14\left(-\frac{1}{2}\right)+1=\mathrm{A}\left(-\frac{1}{2}-1\right)+\mathrm{B}\left(2\left(-\frac{1}{2}\right)+1\right) \Rightarrow -6=-\frac{3}{2}\mathrm{A} \Rightarrow \mathrm{A}=4$

Let $x=1$: $14(1)+1=\mathrm{A}(1-1)+\mathrm{B}(2(1)+1) \Rightarrow 15=3\mathrm{B} \Rightarrow \mathrm{B}=5$

Hence, $\frac{14x+1}{2x^2-x-1}=\frac{4}{2x+1}+\frac{5}{x-1}$

$$\int_0^{\frac{1}{2}}\frac{14x+1}{2x^2-x-1}\,\mathrm{d}x=\int_0^{\frac{1}{2}}\left(\frac{4}{2x+1}+\frac{5}{x-1}\right)\mathrm{d}x$$

$$=\left[2\ln|2x+1|+5\ln|x-1|\right]_0^{\frac{1}{2}}$$

$$=\left[2\ln(2)+5\ln\left(\frac{1}{2}\right)\right]-\left[2\ln(1)+5\ln(1)\right]$$

$$=\ln\left(2^2\right)+\ln\left[\left(\frac{1}{2}\right)^5\right]4=\ln(4)+\ln\left(\frac{1}{32}\right)$$

$$=\ln\left(4\cdot\frac{1}{32}\right)=\ln\left(\frac{1}{8}\right)$$

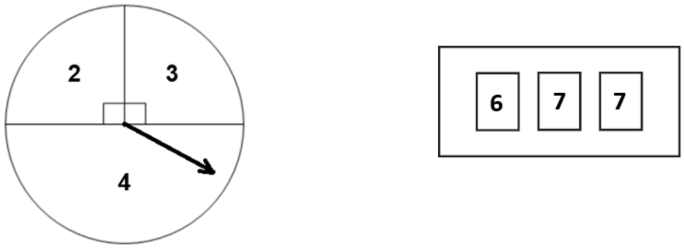
$$=\ln\left(2^{-3}\right)$$

thus, $\int_0^{\frac{1}{2}}\frac{14x+1}{2x^2-x-1}\,\mathrm{d}x=-3\ln(2)$

## Section B (54 marks)

**10.** [Maximum mark: 20]

A spinner consists of an arrow that rotates about the centre of a circle so that one of three numbers is randomly selected (see diagram below). There is also a box containing three numbered cards as shown below. $S$ is the sum of two numbers – one selected randomly with the spinner and the other from randomly selecting one of the cards from the box.




(a) Write down the four different possible values of $S$. [2]

(b) Find the probability of each value of $S$. [5]

(c) Show that the expected value of $S$ is $\frac{119}{12}$. [2]

(d) Anna plays a game where she wins \$15 if $S$ is an even number and loses \$10 if $S$ is an odd number. Sophie plays the game 12 times. Find the amount of money she expects to have at the end of the 12 games. [6]

(e) Anna now plays a different game where she wins \$14 if $S$ is an even number. She loses $x$ dollars if $S$ is an odd number. Find the value of $x$ so that the game is fair. [5]

**Solution**:

(a) possible values of $S$: $\underline{8, 9, 10, 11}$

$8 = 2+6$
$9 = 3+6$ or $2+7$
$10 = 3+7$ or $4+6$
$11 = 4+7$

(b) $P(S=8) = \frac{1}{4}\cdot\frac{1}{3} = \underline{\frac{1}{12}}$

$P(S=9) = \frac{1}{4}\cdot\frac{2}{3} + \frac{1}{4}\cdot\frac{1}{3} = \frac{3}{12} = \underline{\frac{1}{4}}$

$P(S=10) = \frac{1}{4}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{3} = \frac{4}{12} = \underline{\frac{1}{3}}$

$P(S=11) = \frac{1}{2}\cdot\frac{2}{3} = \frac{4}{12} = \underline{\frac{1}{3}}$

(c) $E(S) = \frac{1}{12}(8) + \frac{3}{12}(9) + \frac{4}{12}(10) + \frac{4}{12}(11)$

$= \frac{8}{12} + \frac{27}{12} + \frac{40}{12} + \frac{44}{12} \Rightarrow \underline{E(S) = \frac{119}{12}}$

(d) probability of even number (8 or 10) $= \frac{1}{12} + \frac{4}{12} = \frac{5}{12}$

probability of odd number (9 or 11) $= \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$

expected gain/loss from 1 game $= \frac{5}{12}(15) - \frac{7}{12}(10)$

expected amount of \$ at end of 12 games $= 12\left[\frac{5}{12}(15) - \frac{7}{12}(10)\right]$

$= 5(15) - 7(10) = 75 - 70 = \underline{\$5}$

(e) expected gain/loss from 1 game $= \frac{5}{12}(14) - \frac{7}{12}x = 0 \leftarrow$ 'fair' game

$\frac{70-7x}{12} = 0 \Rightarrow \underline{x = 10}$

**11.** [Maximum mark: 16]

(a) Given that $(x+\mathrm{i}y)^2 = 7-24\mathrm{i}$, $x, y \in \mathbb{R}$, show that

(i) $x^2 - y^2 = 7$;

(ii) $xy = -12$; [4]

(b) Hence, find the two square roots of $7-24\mathrm{i}$ and express them in Cartesian form. [6]

(c) For any complex number, show that $(z^*)^2 = (z^2)^*$. [4]

(d) Hence, write down the two square roots of $7+24\mathrm{i}$ in Cartesian form. [2]

**Solution**:

(a) (i) $(x+\mathrm{i}y)^2 = x^2 + 2\mathrm{i}xy + \mathrm{i}^2y^2 = x^2 - y^2 + 2\mathrm{i}xy = 7-24\mathrm{i}$

Hence, from equating real parts, $x^2 - y^2 = 7$ ***Q.E.D.***

(ii) Equating imaginary parts gives $2\mathrm{i}xy = -24\mathrm{i} \Rightarrow xy = -12$ ***Q.E.D.***

(b) Given that $(x+\mathrm{i}y)^2 = 7-24\mathrm{i}$, the square root of $7-24\mathrm{i}$ is $x+\mathrm{i}y$.

Find $x$ and $y$ by solving the system of equations $\begin{cases} x^2 - y^2 = 7 \\ xy = -12 \end{cases}$

2nd equation gives $y = -\dfrac{12}{x}$; substituting into 1st equation gives $x^2 - \left(-\dfrac{12}{x}\right)^2 = 7$

$x^2 - \dfrac{144}{x^2} = 7 \Rightarrow x^4 - 7x^2 - 144 = 0$ let $x^2 = c$: $c^2 - 7c - 144 = 0 \Rightarrow (c+9)(c-16) = 0$

$x^2 = -9$, $x \in \mathbb{R}$ so not possible; or $x^2 = 16 \Rightarrow x = \pm 4$

If $x = 4$, then $y = -3$; and if $x = -4$, then $y = 3$

Thus, the two square roots of $7-24\mathrm{i}$ are $4-3\mathrm{i}$ and $-4+3\mathrm{i}$

(c) Let $z = a + b\mathrm{i}$; then $z^* = a - b\mathrm{i}$

$(z^*)^2 = (a-b\mathrm{i})^2 = a^2 - 2ab\mathrm{i} + b^2\mathrm{i}^2 = a^2 - b^2 - 2ab\mathrm{i}$

$(z^2)^* = \left[(a+b\mathrm{i})^2\right]^* = \left[a^2 + 2ab\mathrm{i} + b^2\mathrm{i}^2\right]^* = \left[a^2 - b^2 + 2ab\mathrm{i}\right]^* = a^2 - b^2 - 2ab\mathrm{i}$

Thus, for any complex number, $(z^*)^2 = (z^2)^*$ ***Q.E.D.***

(d) Since $(z^*)^2 = (z^2)^*$ then the two square roots of $7+24\mathrm{i}$ are $4+3\mathrm{i}$ and $-4-3\mathrm{i}$

For example, if $z = 4-3\mathrm{i}$ then $(z^2)^* = \left((4-3\mathrm{i})^2\right)^* = (7-24\mathrm{i})^* = 7+24\mathrm{i} = (4+3\mathrm{i})^2 = (z^*)^2$

hence, $4+3\mathrm{i}$ is a square root of $7+24\mathrm{i}$ ; same can be shown for $z = -4-3\mathrm{i}$

**12.** [Maximum mark: 18]

Consider the rational function $f(x)=\dfrac{1}{x-1}$.

(a) The notation $f^{(n)}(x)$ represents the *n*th derivative of *f*. Show that $f^{(3)}(x)=\dfrac{-6}{(x-1)^4}$. [4]

(b) Using mathematical induction, prove that the *n*th derivative of $\dfrac{1}{x-1}$ is $\dfrac{(-1)^n n!}{(x-1)^{n+1}}$. That is,

prove the statement $f^{(n)}(x)=\dfrac{(-1)^n n!}{(x-1)^{n+1}}$ for $n\in\mathbb{Z}^+$ [8]

Consider a different rational function $g(x)=\dfrac{x}{x-1}$.

(c) Prove that the graph of *g* can be produced by translating the graph of *f* one unit vertically in the positive *y* direction. [3]

(d) Hence, write down the formula for the *n*th derivative of *g*. [1]

The graph of *g* is symmetric about the line $y=x$.

(e) Deduce a statement regarding the function $y=g(x)$ and its inverse $y=g^{-1}(x)$. [2]

**Solution**:

(a) $f(x)=\dfrac{1}{x-1}=(x-1)^{-1}$

$f'(x)=-(x-1)^{-2}$; $f^{(2)}(x)=2(x-1)^{-3}$; $f^{(3)}(x)=-6(x-1)^{-4}=\dfrac{-6}{(x-1)^4}$ ***Q.E.D.***

(b) prove the statement (formula) $f^{(n)}(x)=\dfrac{(-1)^n n!}{(x-1)^{n+1}}$ for $n\in\mathbb{Z}^+$

(i) prove statement true for $n=1$:

from (a): $f'(x)=-(x-1)^{-2}=-\dfrac{1}{(x-1)^2}$

substituting $n=1$ into formula: $f^{(1)}(x)=\dfrac{(-1)^1 1!}{(x-1)^{1+1}}=-\dfrac{1}{(x-1)^2}$; hence, statement true for $n=1$

(ii) assume statement is true for $n=k$ where *k* is a specific value of *n*;

that is, assume that $f^{(k)}(x)=\dfrac{(-1)^k k!}{(x-1)^{k+1}}$ is true

(iii) show that from this assumption it must follow that the statement is true for $n=k+1$;

that is, show that $f^{(k+1)}(x)=\dfrac{(-1)^{k+1}(k+1)!}{(x-1)^{k+2}}$

**… continued on next page …**

**12** (b) continued

$$f^{(k+1)}(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left[f^{(k)}(x)\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{(-1)^k k!}{(x-1)^{k+1}}\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[(-1)^k k!(x-1)^{-k-1}\right]$$

$$=-(k+1)(-1)^k k!(x-1)^{-k-2}$$

$$=\left[(k+1)k!\right]\left[(-1)^1(-1)^k\right](x-1)^{-k-2}$$

$$=(k+1)!(-1)^{k+1}(x-1)^{-k-2}$$

$$=\frac{(-1)^{k+1}(k+1)!}{(x-1)^{k+2}} \qquad \textbf{\textit{Q.E.D.}}$$

Hence, the statement is true for $n=k+1$ whenever it is true for $n=k$. The statement is true for $n=1$. Therefore, by the principle of mathematical induction the statement is true for $n\in\mathbb{Z}^+$.

(c) translating the graph of $f$ one unit vertically in the positive $y$ direction:

$$g(x)=f(x)+1=\frac{1}{x-1}+1=\frac{1}{x-1}+\frac{x-1}{x-1}=\frac{x}{x-1} \qquad \textbf{\textit{Q.E.D.}}$$

(d) Since the graph of $g$ is a vertical translation of the graph of $f$ then they have the same gradient for any value of $x$. Thus, the derivatives of $g$ and $f$ must be equal.

Therefore, $g^{(n)}(x)=\dfrac{(-1)^n n!}{(x-1)^{n+1}}$

(e) Since the function $g$ is symmetric about the line $y=x$ then $g$ and its inverse $g^{-1}$ are equal; that is, $g(x)=g^{-1}(x)$. In other words, $g$ is a self-inverse function.